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A General Method for Dynamics Modeling of Flexible Aircraft

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Abstract Equations of motion for flexible aircraft can be developed by applying Lagrange's formulation and the principle of virtual work. These equations involve two different kinds of general coordinates, discrete variables (six degrees-of-freedom motion of the floating coordinates) for ordinary differential equations and distributed variables (displacements) for partial differential equations. Usually, the finite element method is used to discretize the distributed variables, resulting in more unknown variables than the number of equations. Therefore, these equations are not solvable unless reference conditions are defined. Two different methods were developed to solve this problem. One was put forward by David Schmidt using the method of mean-axis. However, this method was questioned by Leonard Meirovitch, who suggested that reference coordinates should be fixed on the material points. In order to prove the correctness of their respective theories, two scholars proposed articles that question each other. In this paper, a more general method is developed by introducing six Lagrange multipliers into equations, and it shows how the constraints are imposed on the reference coordinates. The above two methods can be derived as two special cases by setting special constraints, and are proved both correct. In addition, the general method suggests that the method of mean-axis are more precise and closer to the real situation.

Keywords flexible aircraft · dynamics modeling · equations of motion · mean-axis method · fixed-nodal method · general method

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1 Introduction

Dynamics of unrestrained flexible aircraft were investigated by researchers since decades ago to solve the aeroelasticity problems. Cavin[6] derived the equations of motion and deformation of the flexible body via Hamilton's Principle. He used finete element method to discretize this continuous system and chose mean-axis method to constrain the reference frame assumed that "whatever axis system is used, the small deformation assumptions of the infinitesimal theory of elasticity hold"(page 1686, column 2). There exists at least three method to constrain the reference frame :fixed to structure on an element,fixed relative to the axes of the principle moments of inertia and fixed to mean-axis.

Mean-axis frame consists of six constrains that minimize the kinetic energy of the flexible movement. This method will decouple the rigid-body motion from elastic deformation, greatly simplfy the equations. Mr Schmidt[5][11][12] had done a lot of researches on the control method of the flexible aircraft based on this reference frame .Nodal-fixed method are adopted by Mr Meirovitch[1][2]. He successfully applied this theory to the modeling and control of flexible satellite.

There are also other research groups like Cesnik[14] are doing more in-depth study on nolinear flexible aircraft. Their researches are basically based on these two reference frames, this would not be discussed due to space limitation. In this papaer, a more general method is introduced and concentrated on the diffrences of stiffness matrix between nodal-fixed frame and mean-axis frame.

Equations of motion for flexible aircraft can be developed by applying Lagrange's formulation and the principle of virtual work. A body-reference frame XYZ as shown in Fig. 1 is required firstly to describe the motion of flexible aircraft. This is similar to rigid-body aircraft except that elements on flexible plane will move relatively to the body-reference frame. Without special explanation, aircraft described in the rest of the paper is flexible.

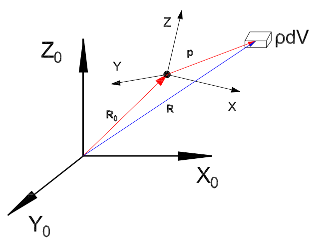

Fig. 1 Element position of flexible aircraft.where $\mathbf{R}$ is the location of the mass element, $\mathbf{R_0}$ stands for the location of reference frame relative to the inertial frame $\mathbf{X_0Y_0Z_0}$, $\mathbf{p_0}$ denotes the location of the mass element of undeformed aircraft relative to the reference frame $\mathbf{XYZ}$, and $\mathbf{u}$ is the deformed displacement of the mass element relative to the undeformed location.

Consider a mass element of the aircraft. Referring to the vector triangle in Fig. 1 gives

$$\mathbf{R} = \mathbf{R_0} + \mathbf{p} = \mathbf{R_0} + \mathbf{p_0} + \mathbf{u} \tag{1}$$

Taking time derivative of Eq. (1) gives the element velocity vector.

$$\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{R_0}}{\mathrm{d}t} + \boldsymbol{\omega} \times \mathbf{p} + \frac{\partial \mathbf{u}}{\partial t} \tag{2}$$

where ω is the angular velocity vector of reference frame $\frac{\mathrm{d}}{\mathrm{d}t}$ means time derivative relative to the inertial frame, $\frac{\partial}{\partial t}$ is the time derivative relative to the reference frame. The kinetic energy T can then be written as

$$T = \frac{1}{2}\int_v \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} \cdot \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} \rho \, dv \tag{3}$$

Representing vectors in Eq. (2) with coordinates gives another expression of the velocity vector, which enables quantitative analysis.

$$\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \mathbf{e}^{bT}\left[C^{br}\dot{R}_0^r + \tilde{\omega}^b\left(p_0^b + u^b\right) + \dot{u}^b\right] \tag{4}$$

where $\mathbf{e}^b$ denotes the reference frame, $\mathbf{e}^r$ denotes the inertial frame,$C^b r$ denotes the direction cosine matrix from $\mathbf{e}^r$ to $\mathbf{e}^b$, the right superscripts '$(\boldsymbol{\cdot})^b$' denotes the coordinates relative to $\mathbf{e}^b$, superscripts '$(\boldsymbol{\cdot})^r$' denotes that we represent the coordinates relative to $\mathbf{e}^r$, over-score '$\tilde{(\cdot)}$ ' denotes a skew-symmetric matrix derived from a 3×1 vector.

The potential energy is effected by gravity and strain.

$$V = V_g + V_e \tag{5}$$

where the gravitational potential V_g is expressed as

$$V_g = -\int_v \mathbf{R} \cdot \mathbf{g}\rho \, dv \tag{6}$$

For the elastic potential energy, its expression for a continuous system is very complicated and is not included here due to space limitation, Please refer to [1] for more information. In the research done by Meirovitch and Schmidt, elastic deformation of the aircraft is described using shape functions.

$$\mathbf{u}\left(x, y, z, t\right) = \sum_{i=1}^{\infty} \boldsymbol{\phi_i}\left(x, y, z\right)\eta_i\left(t\right) \tag{7}$$

where $\boldsymbol{\phi_i}\left(x, y, z\right)$ are space-dependent shape functions and $\Phi = [\boldsymbol{\phi_1} \quad \boldsymbol{\phi_2} \quad \cdots]$ is the united form of $\boldsymbol{\phi_i}\left(x, y, z\right)$, $\eta_i\left(t\right)$ are time-dependent generalized coordinates and $\eta = [\eta_1 \quad \eta_2 \quad \cdots]$ is the united form of $\eta_i\left(t\right)$. Then the continuous system can be discretized and simplified by using Rayleigh-Ritz or Galerkin method to approximate the strain energy,

$$V_e = \frac{1}{2}\eta^T \Phi^T K \Phi \eta \tag{8}$$

where K denotes the stiffness matrix and will be discussed in details later.

The generalized forces needed by Lagrange's formulation are derived by principle of virtual work. It is contributed by distributed aerodynamics and the thrust force of engine.

$$\delta W = \int_v \mathbf{f}_A \cdot \delta \mathbf{R}\, dv + \mathbf{F}_E \cdot \delta \mathbf{R}_E = F^T \delta R_0^b + M^T \delta\theta + \sum_{i=1}^{n} Q_i \delta\eta_i \tag{9}$$

where $\mathbf{f}_A$ denotes the distributed aerodynamics, it should be represented in the body-reference frame. $\mathbf{f}_E$ denotes the thrust force, whose direction actually varies with the elastic displacement but ofen neglected.

Finally, applying the Lagrange's formulation gives the hybrid equations of motion in terms of quasi-coordinates[2], which are used in the Lagrange's formulation when state variables and its derivatives are located in different frames of coordinates.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{R}_0^b}\right) + \tilde{\omega}^b \frac{\partial L}{\partial \dot{R}_0^b} - C^{br}\frac{\partial L}{\partial R_0^r} = F \\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \omega^b}\right) + \tilde{\omega}^b \frac{\partial L}{\partial \omega^b} + \tilde{\dot{R}}_0^b \frac{\partial L}{\partial \dot{R}_0^b} - \left(D^T\right)^{-1}\frac{\partial L}{\partial \theta} = M \\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\eta}}\right) + \frac{\partial L}{\partial \dot{\eta}} = Q
\end{aligned} \tag{10}$$

where the Lagrangian is $L = T - K$. Note that kinetic energy T and potential energy V involve two different kinds of generalized coordinates, discrete variables (R_0^b and ω) for ordinary differential equations and distributed variables u for partial differential equations. Here distributed variables u have been discretised using shape functions.

2 Two special methods

The set of equations in Eqs. (12) cannot be solved because the reference frame is not constrained. It is important to note that the choice of the reference frame in Eq. (1) is arbitrary, which can even be fixed onto the inertial reference frame. However, this has to be done carefully because it might be hard to find a suitable mode for these arbitrarily chosen reference frames due to non-constant stiffness matrices. There are two methods to constrain reference frames, whose frame we choose is defined by the system itself rather than by just giving a motion function.

1)The frame is fixed in the undeformed body, also known as the nodal fixed frame. This type of constrained reference frame is more adopted by researchers. The fixed mass element (reference frame) is treated as a "rigid body" and flies like a rigid-body aircraft, while other elements can be treated as "cantilever beams" clamped at the fixed element. Modes choosen from this type of "bea" usually have higher eigenvalues than free-free mode.

2) The frame is moving relative to the undeformed body, also known as the mean-axis frame[8]. Initially, the origin of mean-axis frame is located at the center of mass of the aircraft. The orientation and displacement of the frame is constrained by the following two conditions.

$$\begin{cases} \int_v \frac{\partial \mathbf{u}}{\partial t} \rho \, dv = 0 \\ \int_v (\mathbf{p_0} + \mathbf{u}) \times \frac{\partial \mathbf{u}}{\partial t} \rho \, dv = 0 \end{cases} \tag{11}$$

These two constraints happen to be satisfied for free-free mode shapes. Without any boundary conditions, the free vibration normal modes of a free beam consist of 6 rigid-body modes and n-6 orthogonal modes, where n denotes infinite dimension. The mean-axis frame can decouple the rigid-body movements and elastic deformations, and is thus more practical from an engineering application point of view. However, Mr Meirovitch questioned this method and claimed it wrong in his article[4](page 503, column 2):

1) "A paradox arises when mean axes are used and the flexibility is modeled by a number of shape functions smaller than or equal to six, as in these cases the number of elastic degrees of freedom is either negative or zero, which is a physical impossibility."

2) "Yet, there is no indication in Refs. 6,7 that the aerodynamic forces were ever transformed from the original body axes to the mean axes, which raises additional doubts about the validity of the formulation."

These two questions were not well answered in the Schmidt's article. He responded that "The shape functions used in Rayleigh-Ritz formulation are the free-vibration normal modes of deformable body" and "Such coupling is obviously present and has been of fundamental interest in the modeling and analysis of the dynamics of rigid atmospheric vehicles since 1904."[5](125501-4,column 2).

Actually, The problem arose when they adopted the mode shapes to reduce the nodal deflection degrees-of-freedom. For a discretized system with n degrees of freedom, the problem described by Eq. (13) is absolutely unsolvable because 6 additional variables ("rigid-body" movement of the reference frame) are introduced when the reference frame is introduced. Only when 6 extra variables are constrained could the equations be solvable. One way is to directly define a motion function for the frame. Another is to finding six boundary conditions to constrain the axis. Essentially, the answer for the first question of Meirovitch is: the 6 "missing" degrees of freedom are expressed as rigid-body rotations and translations.

3 A general method for dynamics modelling of flexible aircraft

The introduction of reference frame results in six more unknown variables than the number of equations. For ease of formula derivation, the Lagrangian formula is expressed as

$$\begin{bmatrix} \hat{I}^T M \hat{I} & -\hat{I}^T M \hat{\tilde{p}}^b & \hat{I}^T M \\ -\hat{\tilde{p}}^{bT} M \hat{I} & -\hat{\tilde{p}}^{bT} M \hat{\tilde{p}}^b & -\hat{\tilde{p}}^{bT} M \\ M \hat{I}^T & -M \hat{\tilde{p}}^b & M \end{bmatrix}_{(3n+6)\times(3n+6)} \begin{bmatrix} \ddot{R}_0^b \\ \dot{\omega}^b \\ \ddot{\hat{u}}^b \end{bmatrix} = \begin{bmatrix} f_R \\ f_\omega \\ f_o \end{bmatrix} \tag{12}$$

note that variables describing the elastic movement are element displacement u^b of n discrete elements, not generalized coordinates η. Regardless of moment of inertia of each element,Suppose the discrete system has 3n translational degrees of freedom. It is important to note that the generalized mass matrices in Eqs. (14) has $3n+6$ rows and $3n+6$ columns, with 6 redundant equations more than it should be. The forces in the above equation are given by

$$\begin{cases} f_R = \hat{I}^T M \hat{\dot{\tilde{u}}}^b \omega^b - \tilde{\omega}^b \left[\hat{I}^T M \hat{I} \dot{R}_0^b - \hat{I}^T M \hat{\tilde{p}}^b \omega^b + \hat{I}^T M \dot{\hat{u}}^b \right] - \hat{I}^T M \hat{I} C^{br} g + F \\ f_\omega = \hat{\dot{\tilde{u}}}^{bT} M \hat{I} \dot{R}_0^b - \left(\hat{\dot{\tilde{u}}}^{bT} M \hat{\tilde{p}}^b + \hat{\tilde{p}}^{bT} M \hat{\dot{\tilde{u}}}^b \right) \omega^b + \hat{\dot{\tilde{u}}}^{bT} M \dot{\hat{u}}^b - \tilde{\omega}^b \left(-\hat{\tilde{p}}^{bT} M \hat{I} \dot{R}_0^b \right. \\ \qquad \left. + \hat{\tilde{p}}^{bT} M \hat{\tilde{p}}^b \omega^b - \hat{\tilde{p}}^{bT} M \dot{\hat{u}}^b \right) - \dot{\tilde{R}}_0^b \left(\hat{I}^T M \hat{I} \dot{R}_0^b - \hat{I}^T M \hat{\tilde{p}}^b \omega^b + \hat{I}^T M \dot{\hat{u}}^b \right) + M \\ f_o = M \hat{\dot{\tilde{u}}}^b \omega^b - K \hat{u}^b + \dfrac{\mathrm{d}T}{\mathrm{d}\hat{u}} + Q \end{cases} \tag{13}$$

where over-score '$\hat{\cdot}$' denotes a united form of element information. For example, $\hat{I} = \left[I \cdots I \right]^T_{3n\times3}$ is a united form of 3-order identity matrix, $\hat{\tilde{p}}^b = \left[\tilde{p}_1^b \cdots \tilde{p}_n^b \right]^T$, and $\hat{u}^b = \left[u_1^b \cdots u_n^b \right]^T_{3n\times1}$. M is a mass matrix, $M = \begin{bmatrix} m_1 I & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & m_n I \end{bmatrix}$

By introducing six Lagrange multipliers into equations, the constraints are imposed on the reference coordinates. Constrains applied by the reference frame can be written in a uniform expressions.

$$H(q,t) = 0 \tag{14}$$

where $q = (q_1 \quad q_2 \quad \cdots q_s)$ is the generalised coordinates vector, and t is included if constrains are related to time. According to virtual displacement principle, the uniform expression of constrains equations for speed and acceleration are :

$$\begin{aligned} \dot{H} &= H_q \dot{q} + H_t = 0 \\ H_q \ddot{q} &= -\left[(H_q \dot{q})_q + H_{qt} \dot{q} + H_{tq} \dot{q} + H_{tt} \right] \end{aligned} \tag{15}$$

where $\gamma = (H_q \dot{q})_q + H_{qt} \dot{q} + H_{tq} \dot{q} + H_{tt}$. Combine the Eqs. (17) with the Eqs. (14), and the augmented formula equation is

$$\begin{bmatrix} \hat{I}^T M \hat{I} & -\hat{I}^T M \hat{\tilde{p}}^b & \hat{I}^T M & 0 \\ -\hat{\tilde{p}}^{bT} M \hat{I} & -\hat{\tilde{p}}^{bT} M \hat{\tilde{p}}^b & -\hat{\tilde{p}}^{bT} M & 0 \\ M \hat{I}^T & -M \hat{\tilde{p}}^b & M & H_{\hat{u}}^T \\ 0 & 0 & H_{\hat{u}} & 0 \end{bmatrix}_{(3n+12)\times(3n+12)} \begin{bmatrix} \ddot{R}_0^b \\ \dot{\omega}^b \\ \ddot{\hat{u}}^b \\ \mu \end{bmatrix} = \begin{bmatrix} f_R \\ f_\omega \\ f_o \\ \gamma \end{bmatrix} \tag{16}$$

where $H_q = \left[\mathbf{0}_{6\times 6}\ H_{\hat{u}}^T\right]$. Consider the third row in the above matrix equation.,

$$M\hat{I}\ddot{R}_0^b - M\hat{\tilde{p}}^b\dot{\omega}^b + M\hat{\ddot{u}}^b + H_{\hat{u}}^T\mu = f_o \tag{17}$$

Then element acceleration $\hat{\ddot{u}}^b$ can be expressed. Substituting $\hat{\ddot{u}}^b$ into constrains Eqs. (17), and extracting μ give

$$\mu = \left(H_{\hat{u}}M^{-1}H_{\hat{u}}^T\right)^{-1}\left[H_{\hat{u}}M^{-1}\left(f_o - M\hat{I}\ddot{R}_0^b + M\hat{\tilde{p}}^b\dot{\omega}^b\right) - \gamma\right] \tag{18}$$

Set parameters K' and K''

$$\begin{aligned} K' &= H_{\hat{u}}^T\left(H_{\hat{u}}M^{-1}H_{\hat{u}}^T\right)^{-1}H_{\hat{u}}M^{-1} \\ K'' &= H_{\hat{u}}^T\left(H_{\hat{u}}M^{-1}H_{\hat{u}}^T\right)^{-1} \end{aligned} \tag{19}$$

Insert μ back to the Eq. (20), and the third row would be reduced to

$$M\hat{I}\ddot{R}_0^b - M\hat{\tilde{p}}^b\dot{\omega}^b + M\hat{\ddot{u}}^b + K'\left[\left(f_o - M\hat{I}\ddot{R}_0^b + M\hat{\tilde{p}}^b\dot{\omega}^b\right)\right] - K''\gamma = f_o \tag{20}$$

Rebuild the Eq. (18)

$$\begin{bmatrix} \hat{I}^T M\hat{I} & -\hat{I}^T M\hat{\tilde{p}}^b & \hat{I}^T M \\ -\hat{\tilde{p}}^{bT}M\hat{I} & -\hat{\tilde{p}}^{bT}M\hat{\tilde{p}}^b & -\hat{\tilde{p}}^{bT}M \\ \left(\bar{I}-K'\right)M\hat{I}^T & -\left(\bar{I}-K'\right)M\hat{\tilde{p}}^b & M \end{bmatrix}_{(3n+6)\times(3n+6)} \begin{bmatrix} \ddot{R}_0^b \\ \dot{\omega}^b \\ \hat{\ddot{u}}^b \end{bmatrix} = \begin{bmatrix} f_R \\ f_\omega \\ \left(\bar{I}-K'\right)f_o + K''\gamma \end{bmatrix} \tag{21}$$

where $\bar{I}$ denotes Identity matrix of order $3n$. Eq. (21) seems also unsolvable. However, elastic deformation $\hat{u}^b$ has only $3n-6$ degrees-of-freedom and can be solved by left multiply the matirx $\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \Phi'^T \end{bmatrix}$, where Φ' consists of $3n-6$ shape functions(not modes) when $\hat{u}^b = \Phi'\eta$.

3.1 Nodal fixed reference frame

Firstly, locate the reference frame on an element **o**. The displacement of element **o** relative to reference frame should be zero. Then, describe the orientation of the frame with another two elements **j** and **k**. Let x axis cross the element **j** and keep the element **k** always on the x-y plane.

$$u_o = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^T, u_j = \begin{bmatrix}x & 0 & 0\end{bmatrix}^T, u_k = \begin{bmatrix}x & y & 0\end{bmatrix}^T \tag{22}$$

They are 6 constrain equations,expressed then as the form in Eq. (14).

$$H_{\hat{u}} = \begin{bmatrix} I_{6\times 6} \\ 0_{(3n-6)\times 6} \end{bmatrix}^T = 0 \tag{23}$$

Insert Eq. (23) back to K' and eliminate the K'' when $\gamma = 0$.

$$K' = H_{\hat{u}}^T \left(H_{\hat{u}} M^{-1} H_{\hat{u}}^T\right)^{-1} H_{\hat{u}} M^{-1} = \begin{bmatrix} I_{6\times 6} & 0 \\ 0 & 0 \end{bmatrix}_{3n\times 3n} \tag{24}$$

Ignoring the effect of dynamic stiffening[13], the third row in Eq. (16) can be written as the form of block matrix.

$$\begin{bmatrix} M_{mm} & M_{mn} \\ M_{nm} & M_{nn} \end{bmatrix} \begin{bmatrix} \hat{\ddot{u}}_m \\ \hat{\ddot{u}}_n \end{bmatrix} + \begin{bmatrix} 0_{6\times 6} & 0 \\ 0 & I_{(3n-6)\times(3n-6)} \end{bmatrix} K \begin{bmatrix} \hat{u}_m \\ \hat{u}_n \end{bmatrix} = \begin{bmatrix} Q_m \\ Q_n \end{bmatrix} \tag{25}$$

where $\hat{u}_m = [u_{ox} \quad u_{oy} \quad u_{oz} \quad u_{jy} \quad u_{jz} \quad u_{kx}] = 0$ denotes the 6 fixed degrees-of-freedom, $\hat{u}_n$ denotes the n-6 remaining unconstrained degrees-of-freedom.

$$M_{nn} \hat{\ddot{u}}_n + K_{nn} \hat{u}_n = Q_n \tag{26}$$

Eq. (26) implies that the order of Eq. (12) with 6 additional degrees-of-freedom is reduced to $3n$, leading to a solvable equations.

3.2 Mean-axis reference frame

Express the constrains of mean-axis frame described in Eqs. (14) into matrix form :

$$\begin{cases} \hat{I}^T M \hat{u}^b = 0 \\ \hat{\tilde{p}}_0^T M \hat{u}^b = 0 \\ \hat{I}^T M \hat{\tilde{p}}_0^b = 0 \end{cases} \Leftrightarrow \begin{cases} \hat{I}^T M \hat{\dot{u}}^b = 0 \\ \hat{\tilde{p}}_0^T M \hat{\dot{u}}^b = 0 \\ \hat{I}^T M \left(\hat{\tilde{p}}_0^b + \hat{u}^b\right) = 0 \end{cases} \Rightarrow H_{\hat{u}} = \begin{bmatrix} \hat{I}^T M \\ \hat{\tilde{p}}_0^{bT} M \end{bmatrix} \tag{27}$$

Insert Eqs. (27) into Eqs. (16), some coupled terms in generalized mass matrix would be decoupled.

$$\begin{bmatrix} \hat{I}^T M \hat{I} & 0 & 0 \\ 0 & -\hat{\tilde{p}}^{bT} M \hat{\tilde{p}}^b & 0 \\ \left(\bar{I} - K'\right) M \hat{I}^T & -\left(\bar{I} - K'\right) M \hat{\tilde{p}}^b & M \end{bmatrix} \begin{bmatrix} \ddot{R}_0^b \\ \dot{\omega}^b \\ \hat{\ddot{u}}^b \end{bmatrix} = \begin{bmatrix} f_R \\ f_\omega \\ \left(\bar{I} - K'\right) f_o \end{bmatrix} \tag{28}$$

where

$$K' = M \left[\hat{I} \ \hat{\tilde{p}}_0^b\right] \begin{bmatrix} mI & 0 \\ 0 & \hat{\tilde{p}}_0^{bT} M \hat{\tilde{p}}_0^b \end{bmatrix} \begin{bmatrix} \hat{I}^T \\ \hat{\tilde{p}}_0^{bT} \end{bmatrix} \tag{29}$$

Ignoring the effect of dynamic stiffening, the third row in Eq. (16) can be written as

$$M \hat{\ddot{u}}^b + \left(\bar{I} - K'\right) K \hat{u}^b = \left(\bar{I} - K'\right) Q \tag{30}$$

modes of the Free-vibration are used as shape functions to find the modes associated with mean-axis frame. Left multiply both sides of the Eq. (28) by matirx $\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \Phi'^T \end{bmatrix}$,then we get the result suggested by Schmidt.

$$\begin{bmatrix} \hat{I}^T M \hat{I} & 0 & 0 \\ 0 & -\hat{\tilde{p}}^{bT} M \hat{\tilde{p}}^b & 0 \\ 0 & 0 & \Phi'^T M \Phi' \end{bmatrix} \begin{bmatrix} \ddot{R}_0^b \\ \dot{\omega}^b \\ \eta \end{bmatrix} = \begin{bmatrix} f_R \\ f_\omega \\ \Phi'^T f_o \end{bmatrix} \tag{31}$$

4 discussion

There exists only one truth for a real motion of an aircraft whatever reference frame we adopt. Suppose that both two reference frame used by Schmidt and Meirovitch are reasonable, then the kinetic energy and potential energy corresponding to each other should be the same. Let $\mathbf{R}_{real}$ be the real motion of the aircraft and describe this motion using reference frame $\mathbf{e}^{b1}$ and $\mathbf{e}^{b2}$ respectively.

$$\mathbf{R}_{real} = \mathbf{R}_1 + \mathbf{p}_1 + \mathbf{u}_1 = \mathbf{R}_2 + \mathbf{p}_2 + \mathbf{u}_2 \tag{32}$$

$\mathbf{u}_2$ could be represented by $\mathbf{u}_1$. The potential energy due to elastic deformation in reference frame $\mathbf{e}^{b1}$ could be written as

$$V_e = \frac{1}{2}\hat{u}^{b_2 T} K^{b_2} \hat{u}^{b_2} = \frac{1}{2}\hat{u}^{b_1 T}\left[\hat{C}^{b_1 b_2} K^{b_2} \hat{C}^{b_2 b_1}\right]\hat{u}^{b_1} \tag{33}$$

where $\hat{C}^{b_1 b_2}$ is united form of direction cosine matrix $C^{b_1 b_2}$ between frame $\mathbf{e}^{b_1}$ and $\mathbf{e}^{b_2}$. Stiffness matrix K would vary with time when relative motion of these two frames exists. Only one type of reference frame makes its corresponding stiffness matrix a constant. call this type of frame the based reference frame.

Firstly discuss the nodal-fixed frames in Fig. (2), in which $\mathbf{e}^{b_1}$ is rotating relative to $\mathbf{e}^{b_2}$. It is hard to find a nodal-fixed reference frame that let stiffness matrix K happen to be a constant.

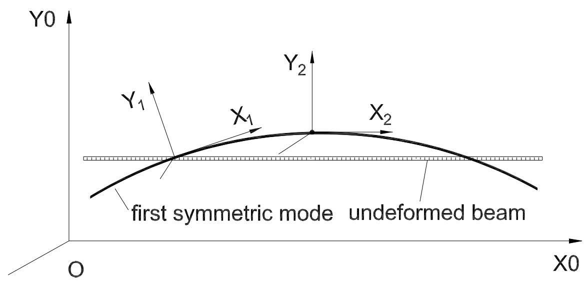

Fig. 2 nodal-fixed frames fixed on different elements of the free beam. Frame $\mathbf{e}^{b_1}$ is fixed to somewhere of the beam and $\mathbf{e}^{b_2}$ is fixed to middle. $\mathbf{e}^{b_1}$ is rotating relative to $\mathbf{e}^{b_2}$.

Another requirement is that the eigenvalues shall not change whatever reference frame we adopt. From this, Mean-axis is the true reference frame for a free structure. However, given the condition that the beam is fixed at one end, it is nodal-fixed frame that let K be a constant.

Boundary conditions will change the structure's stiffness when forces coupled with the structure and feedback to structure itself, $Q_m = -K_{mn}\hat{u}_n$ for example. It is hard to say which frame and modes are right because aerodynamics can also be expressed like feedback forces.

5 Conclusion

A more general method is introduced in this paper. This method is based on the application of six Lagrange multipliers into equations. Reference frame could be settled once when six constrains are confirmed, mean-axis and nodal-fixed frame are merely special cases of the general method. More important is that only one type of reference frame has a constant stiffness matrix K. Others who rotate relative to it will have a time-varing matrix K. Mean-axis is true axis when structure is free, and nodal-fixed frame is suitable when structure is fixed on wall like a cantiver beam. For more accurate modeling of aircraft, one way is to take into account the stiffness effected by aerodynamics firstly and then find the based reference frame in turn. The changed stiffness matrix is $(\bar{I} - K')(\underline{I} - \underline{K})K$ where $(\underline{I} - \underline{K})$ denotes the change of matrix K due to the aerodynamics. Whatever, the mean-axis is closer to the truth. Because a real flexible aircraft won't always be mounted on the bracket in wind tunnel.

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